Self-dual codes which are principal ideals of the group algebra "F2[F2m,+]"

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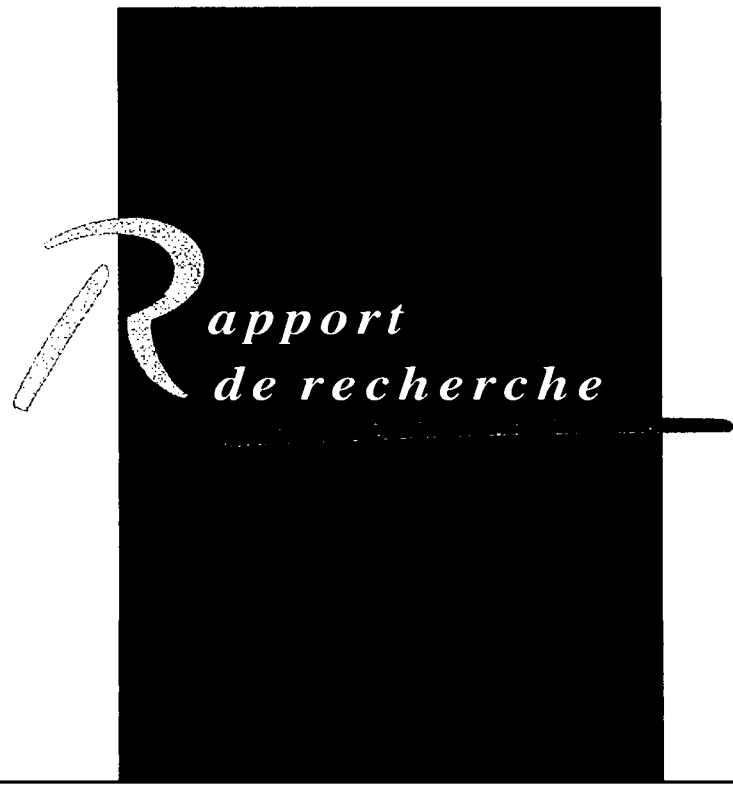

Codes principaux autoduaux de l'algèbre de groupe $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$
Self-dual codes which are principal ideals of the group algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$ *

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Résumé

Cet article est un prolongement du travail de Paul CAMION portant sur les codes autoduaux, idéaux principaux de l'algèbre $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$. Dans [5], ces codes sont introduits et appelés H-codes. Notre principal résultat ici est que les H-codes sont aymptotiquement mauvais, au sens de la borne de Gilbert-Varshamov. Pour obtenir ce résultat, nous exhibons d'abord une borne supérieure pour la distance minimale de tout H-code donné. Nous caractérisons les H-codes extrémaux et montrons que leurs générateurs sont liés à certains ensembles à différence.

Abstract

This paper follows up CAMION's contribution on self-dual codes which are principal ideals of the algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$, the so-called H-codes. Our main result is that this class of codes does not meet the Gilbert-Varshamov bound. We obtain this result by giving an upper bound on the minimal distance of any H-code. We characterize extremal H-codes and link up their generators with certain difference sets.

Keywords: self-dual codes, extremal codes, group algebra, difference sets.

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Self-dual codes which are principal ideals of the group algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$

1 Introduction

Let C be a binary linear code of length n. The code C is said to be *self-dual* if and only if it is equal to its dual; in this case its dimension is $n/2$. A code is said to be t-divisible when the distance between codewords is always a multiple of t. By convention a t-divisible code is not t'-divisible for $t' > t$.

Our main reference on self-dual codes is [17, chapter 19], where an extensive study of the work of GLEASON is given. A binary t-divisible self-dual code exists for only two values of t: $t = 2$ or 4. When $t = 2$ we say that the code is *even*; when $t = 4$ we say that the code is ***doubly-even***. An important result on these self-dual codes was obtained by GLEASON when he established strong constraints on their weight enumerators [14]. Then an upper bound on the minimal distance of these codes appeared; a code which meets this bound is said to be extremal. Many questions on the existence of extremal self-dual codes remain open.

Binary H-codes were introduced by CAMION in [5]. They form a class of binary self-dual codes which are principal ideals of a modular algebra of type $\mathbb{F}_2[G]$, $G = \{\mathbb{F}_{2^m}, +\}$. Basic properties of such codes are given in [5]. Above all it is shown that it is very easy to construct an H-code; it is sufficient to choose a hyperplane H of G and a codeword x of odd weight whose support is contained in H. A generator matrix is directly obtained. The code is even if and only if the weight of x is congruent to 1 modulo 4; otherwise the code is doubly-even. It is also shown in [5] that there are many extremal doubly-even H-codes of length 32. In fact many will be equivalent because there are only five extremal codes of length 32 [13, 15]. Our purpose in this paper is to study asymptotic properties of the minimal distance of H-codes.

We will denote by $\mathcal{A}$ the algebra $\mathbb{F}_2[G]$. In Section 2, we present the class of H-codes, giving a brief review of properties of the algebra $\mathcal{A}$; more details on codes of $\mathcal{A}$ can be found in [2, 8, 11]. Recall that the powers of the radical of $\mathcal{A}$ are the Reed-Muller codes [4].

Extremal H-codes are studied in Section 3. We first give an upper bound on the minimal distance of any H-code (Theorem 3). We can then deduce that H-codes are asymptotically bad; moreover we prove that there are no extremal H-codes of length greater than 32.

The last section treats the extremal H-codes of length 32. Our main result is presented in Theorem 5, linking extremal H-codes to difference sets of the abelian group $\{\mathbb{F}_{16}, +\}$.

The usual terminology of coding theory can be found in [17]. In this paper we only treat binary linear codes of length 2^m. The distance is always the Hamming distance.

2 The class of H-codes

2.1 The algebra $\mathcal{A}$

Let G be the additive group of the finite field $\mathbb{F}_{2^m}$. We denote by $\mathcal{A}$ the algebra of the group G over the field $\mathbb{F}_2$. An element of $\mathcal{A}$ is a formal sum

$$\mathrm{x} = \sum_{g \in G} x_g X^g, \; x_g \in \mathbb{F}_2 \; .$$

For any $a \in \mathbb{F}_2$, $b \in \mathbb{F}_2$, $\mathrm{x} \in \mathcal{A}$ and $\mathrm{y} \in \mathcal{A}$, we have:

$$ax + by = a \sum_{g \in G} x_g X^g + b \sum_{g \in G} y_g X^g = \sum_{g \in G} (a x_g + b y_g) X^g \quad ,$$

$$\mathrm{xy} = \sum_{g \in G} x_g X^g \sum_{g \in G} y_g X^g = \sum_{h \in G} \left(\sum_{g \in G} x_g y_{h+g} \right) X^h \quad ,$$

The additive and multiplicative unit elements are $0 = \sum_{g \in G} 0 \; X^g$ and $1 = X^0$. In this paper, the algebra $\mathcal{A}$ is the ambient space of the codes. A code of $\mathcal{A}$ is an $\mathbb{F}_2$-subspace. A codeword is an element of $\mathcal{A}$. Note that an ideal of $\mathcal{A}$ is a code invariant under the multiplication by X^g, for any $g \in G$. The scalar product is $< \mathrm{x}, \mathrm{y} >= \sum_{g \in G} x_g y_g$. The dual of a code C will be denoted by $C^\perp$. If C is an ideal, the annihilator of C is:

$$Ann \; C = \{ \; \mathrm{x} \in \mathcal{A} \mid \mathrm{xy} = 0 \;\; \text{for all} \;\; \mathrm{y} \in C \; \} \; .$$

Since $\mathrm{xy} = \sum_{h \in G} < \mathrm{x}, X^h \mathrm{y} > X^h$, the following proposition is obvious:

Proposition 1 *Let C be an ideal of $\mathcal{A}$. Then $Ann \; C = C^\perp$.*

Since G is an elementary abelian 2-group and $a^2 = a$ for $a \in \mathbb{F}_2$. we have for any element x of $\mathcal{A}$:

$$\mathrm{x}^2 = \left(\sum_{g \in G} x_g X^g \right)^2 = \sum_{g \in G} x_g X^{2g} = \left(\sum_{g \in G} x_g \right) X^0 \; .$$

That means that x is invertible in $\mathcal{A}$ if and only if its weight is odd. The set of non invertible codewords is the only maximal ideal of $\mathcal{A}$, its so-called *radical*. We will denote by $\mathcal{P}$ the radical of $\mathcal{A}$; that is

$$\mathcal{P} = \{ \; \mathrm{x} \in \mathcal{A} \mid \sum_{g \in G} x_g = 0 \; \} = \{ \; \mathrm{x} \in \mathcal{A} \mid \mathrm{x}^2 = 0 \; \} \; .$$

For any $j \in [1, m]$ we define the ideal which is the j-power of $\mathcal{P}$. The ideal $\mathcal{P}^j$ is generated linearly by elements of the form $\prod_{i=1}^{j} \mathrm{x}_i$ where $\mathrm{x}_i \in \mathcal{P}$. For any basis $\{e_1, \ldots, e_m\}$ of G, the 2^m elements

$$\left\{ \prod_{i=1}^{m} (X^{e_i} + 1)^{\lambda_i} \mid \lambda_i \in \{0, 1\} \right\}$$

form a linear basis for $\mathcal{A}$. Moreover, for all $j \in [1,m]$, the elements of

$$\mathcal{B}^j = \left\{ \prod_{i=1}^{m}(X^{e_i}+1)^{\lambda_i} \mid j \leq \sum_{i=1}^{m}\lambda_i \ , \ \lambda_i \in \{0,1\} \right\} \tag{1}$$

form a linear basis of $\mathcal{P}^j$ (by convention, $\mathcal{P}^0 = \mathcal{A}$) [8]. We now define a useful parameter of an element or of a subset of $\mathcal{A}$:

Definition 1 *Let* $\mathrm{x} \in \mathcal{A}$ *; let* U *be a subset of* $\mathcal{A}$. *Let* $j \in [0,m]$.

- *We will say that the depth of* x *equals* j *if and only if* x *is in* $\mathcal{P}^j$ *and not in* $\mathcal{P}^{j+1}$.
- *We will say that the depth of* U *equals* j *if and only if* U *is included in* $\mathcal{P}^j$ *and not included in* $\mathcal{P}^{j+1}$.

In the following we will use several representations of codewords of depth 1. Two of them are given in Lemma 1. It is clear, by means of the basis $\mathcal{B}^0$, that a codeword z of depth 0 is of the form:

$$\mathrm{z} = X^0 + \mathrm{z}' = 1 + \mathrm{z}' \ , \quad \mathrm{z}' \in \mathcal{P} \ . \tag{2}$$

Lemma 1 *Let* $\mathrm{x} \in \mathcal{A}$ *be a codeword of depth* 1. *By using the basis* $\mathcal{B}^1$ *(cf. (1)), we have:*

$$\mathrm{x} = \sum_{i=1}^{m} a_i(X^{e_i}+1) + \mathrm{y} \ , \quad \mathrm{y} \in \mathcal{P}^2 \ , \quad a_i \in \mathbb{F}_2 \ , \tag{3}$$

where at least one a_i *is not zero. Set* $h = \sum_{i=1}^{m} a_i e_i$. *Then*

$$\mathrm{x} = (X^h+1) + \mathrm{x}' \ , \quad \mathrm{x}' \in \mathcal{P}^2 \ . \tag{4}$$

Proof: We can assume that $a_1 \neq 0$ in (3). Since $e_1 = h + \sum_{i=2}^{m} a_i e_i$, we have

$$\begin{aligned} X^{e_1}+1 &= X^h \prod_{i=2}^{m} X^{a_i e_i} + 1 = ((X^h+1)+1)\left(\prod_{i=2}^{m}((X^{e_i}+1)^{a_i}+1)\right)+1 \\ &= (X^h+1) + \sum_{i=2}^{m} a_i(X^{e_i}+1) + \mathrm{y} \ , \quad \mathrm{y} \in \mathcal{P}^2 \ . \end{aligned}$$

Then, according to (3), $\mathrm{x} = (X^h+1) + \mathrm{y}$.
□

BERMAN proved in [4] that the powers of the radical of $\mathcal{A}$ are the Reed-Muller codes (RM-codes). More precisely:

$$\mathcal{P}^j = R(m-j,m) \quad , \text{ for all } j \in [1,m] \ , \tag{5}$$

where $R(m-j,m)$ is the RM-code of length 2^m and order $m-j$ (see an easy proof in [1]).

Remark 1: From well-known properties of RM-codes or by using the bases $\mathcal{B}^j$, it follows easily that :

1. $\mathcal{P}^{m+1} = \{0\}$ and $\mathcal{P}^m = \{ a \sum_{g \in G} X^g \mid a \in \mathbb{F}_2 \}$;
2. The minimal distance of $\mathcal{P}^j$ equals 2^j.
3. $(\mathcal{P}^j)^\perp = Ann\ \mathcal{P}^j = \mathcal{P}^{m-j+1}$;
4. Since $\mathcal{P}^{m-1}$ is the RM-code of order 1, the set $\mathcal{P}^{m-1} \backslash \mathcal{P}^m$ contains the elements of $\mathcal{A}$ whose supports are the affine hyperplanes of G ;
5. Any ideal of $\mathcal{A}$ of depth j contains $\mathcal{P}^m$ and is such that its intersection with $\mathcal{P}^k$, $k \geq j$ is not empty.

The *support* of a codeword $\mathrm{x} = \sum_{g \in G} x_g X^g$ is:

$$supp(\mathrm{x}) = \{ g \in G \mid x_g = 1 \} .$$

Recall that the cardinal of $supp(\mathrm{x})$ is the *weight* of x, which we will denote by $\omega(\mathrm{x})$. We will use also the notation $\mathbb{F}_2 G$ instead of $\mathcal{A}$. More generally the notation $\mathbb{F}_2 V$, where V is an $\mathbb{F}_2$-subspace of G, will appear. It is because we will often need to study some codeword whose support is contained in some V. For instance the *restriction* of x to V is $\mathrm{x}' = \sum_{g \in V} x_g X^g$ and we can identify it with a codeword of the algebra $\mathbb{F}_2 V$. The following lemma will be very useful for the proof of Theorem 3.

Lemma 2 *Let V be a subspace of G of dimension k. Let W be a subset of G of 2^{m-k} elements, which contains one representative of each coset of V : W is such that G is equal to $\cup_{h \in W}(h + V)$. Let* x *in* $\mathcal{A}$ *and* $\mathrm{y} = \sum_{g \in V} X^g$. *Let us denote by* $\mathrm{x}(h)$, $h \in W$, *the restriction of* $X^h \mathrm{x}$ *to* V. *So* x *can be written as follows :*

$$\mathrm{x} = \sum_{h \in W} \mathrm{x}(h) X^h \quad , \quad \mathrm{x}(h) \in \mathbb{F}_2 V .$$

Then either $\mathrm{x}(h)$ *is invertible and the restriction of* yx *to* $h + V$ *equals* $\mathrm{y} X^h$ *or* $\mathrm{x}(h) \in \mathcal{P}$ *and the restriction of* yx *to* $h + V$ *equals* 0. *The weight of* yx *equals* $\lambda 2^k$, *where* λ *is the number of* $h \in W$ *such that* $\mathrm{x}(h)$ *is invertible.*

Proof. The codeword y can be considered as an element of $\mathbb{F}_2 V$; it is the all-one vector of $\mathbb{F}_2 V$. Note that, for any basis $(e_1, \ldots, e_k)$ of V, y equals $\prod_{i=1}^{k} (X^{e_i} + 1)$. So $\mathrm{y} \in \mathcal{P}^k$ and $\mathcal{P}^{k+1} \cap \mathbb{F}_2 V = \{0\}$. We have

$$\mathrm{yx} = \sum_{h \in W} \mathrm{yx}(h) X^h \quad , \quad \mathrm{yx}(h) \in \mathcal{P}^k \cap \mathbb{F}_2 V .$$

Whenever $\mathrm{x}(h) \in \mathcal{P}$ the product $\mathrm{yx}(h)$ is zero, since its depth is $k + 1$. If $\mathrm{x}(h)$ is invertible, then $\mathrm{x}(h) = 1 + \mathrm{z}$, $\mathrm{z} \in \mathcal{P}$, which yields $\mathrm{yx}(h) = \mathrm{y}$.
□

2.2 Principal ideals of depth 1

The definition of the H-codes and the properties given in this section are mainly due to CAMION [5]. We denote by (x) the principal ideal generated by an element x of $\mathcal{A}$. That is

$$(x) = \{ yx \mid y \in \mathcal{A} \} .$$

A generator of (x) is an element yx such that y has depth 0 (i.e. is invertible); all generators have the same depth. Since the ideal (x) equals (yx) for all such y, we always will consider x as a formal generator that we can fix when it is necessary. In accordance with Proposition 1, an element z is in the dual of the ideal (x) if and only if it is in its annihilator. So

$$(x)^{\perp} = Ann\ (x) = \{ z \in \mathcal{A} \mid zx = 0 \} .$$

The algebra $\mathcal{A}$ itself is the only principal ideal of depth 0. When $x \in \mathcal{P}$, then $x^2 = 0$ which yields that the ideal (x) is contained in its dual ; the dimension of (x) is less than or equal to 2^{m-1}.

From now on we assume that x has depth 1 - i.e. $x \in \mathcal{P} \backslash \mathcal{P}^2$. We assume that x is written as in (3); we can suppose, without loss of generality, that $a_1 = 1$. Hence the restriction of x to the hyperplane H generated by $(e_2, ..., e_m)$ is invertible. Indeed set $z = \sum_{g \in H} X^g$; using (3), with $a_1 = 1$, we obtain:

$$zx = \prod_{i=2}^{m} (X^{e_i} + 1)\ x = \prod_{i=1}^{m} (X^{e_i} + 1) = \sum_{g \in G} X^g ,$$

which implies, with the notation of Lemma 2, that

$$x = x(0)X^0 + x(h)X^h \quad , \quad h \notin H ,$$

where x(0) and x(h) are invertible elements of $\mathbb{F}_2 H$. Note that if $x \in \mathcal{P}^2$ the product zx is zero for any hyperplane H. By multiplying x by x(0), we obtain a generator of (x) of the form

$$X^0 + x'X^h \quad , \quad h \notin H \ , \ x' \text{ invertible} \ , \ x' \in \mathbb{F}_2 H . \tag{6}$$

Now it is clear that the set $\{ X^g + (x'X^g)X^h \mid g \in H \}$ is a linearly independent subset of (x) containing 2^{m-1} elements. Since $dim\ (x) \leq 2^{m-1}$, it is a basis of (x). More generally the set $\{ X^g x \mid g \in H \}$ it is a basis of (x). Then we have proved the following theorem.

Theorem 1 *Let* x *in* $\mathcal{A}$. *Recall that* $\mathcal{A} = \mathbb{F}_2 G$ *where* G *is the additive group of* $\mathbb{F}_{2^m}$.

(i) *The element* x *is of depth* 1 *if and only if there exists a hyperplane* H *of* G *such that the restriction of* x *to* H *(respectively to* $h + H$, $h \notin H$*) is invertible.*

(ii) *Assume that* x *is of depth* 1. *Then the ideal* (x) *is self-dual. Its dimension equals* 2^{m-1}. *The code* (x) *is said to be an* H*-code, where* H *is a hyperplane of* G *such that*

$$x = x' + x''X^h \quad , \quad h \notin H \ , \ supp(x') \subset H \ , \ supp(x'') \subset H \ ,$$

where x′ *and* x″ *are invertible. The set* $\{ X^g \mathrm{x} \mid g \in H \}$ *is a basis of* (x). *The generators of* (x) *are the*

$$\sum_{g \in H} a_g \, X^g \mathrm{x} \ , \quad a_g \in \mathbb{F}_2, \quad \textit{where} \quad \sum_{g \in H} a_g X^g \quad \textit{is invertible} \ .$$

There is a generator of (x) *of the form (6).*

The H-codes are the principal ideals of depth 1.

Remark 2: Principal ideals of depth 1 of the algebra $K[G]$, $K = \mathbb{F}_{2^r}$ and $G = \mathbb{F}_{2^m}$, are self-dual. WOLFMANN proved that the binary image of such codes, relative to a convenient basis, are binary self-dual codes, which can be doubly-even [19]. For instance an extremal $[64, 32, 12]$ code was constructed in this way [18].

An H-code (x) is contained in $\mathcal{P}$; thus it is an even self-dual code. Moreover (x) has a generator matrix all of whose rows have the same weight as x. So it is easy to determine if the code is doubly-even or not, since a binary self-dual code is doubly-even if and only if every row of its generator matrix has a weight divisible by 4.

Corollary 1 *Let* x *be an element of* $\mathcal{A}$ *of depth* 1. *Then the principal ideal* (x), *generated by* x, *is an even self-dual code. Moreover* (x) *is doubly-even if and only if the weight of* x *is divisible by* 4.

Constructing a doubly-even self-dual code of length 2^m. We can summarize as follows the construction of the generator matrix of an H-code of length 2^m:

- Choose a hyperplane H of the $\mathbb{F}_2$-vector space $\mathbb{F}_{2^m}$.
- Get a codeword x′ of odd weight whose support is contained in H. That means that x′ is an invertible element of $\mathbb{F}_2 H$.
- Compute the $2^{m-1} \times 2^{m-1}$ matrix M whose rows are the codewords $X^h \mathrm{x}'$, $h \in H$.

Let I be the $2^{m-1} \times 2^{m-1}$ identity matrix. Then the code with generator matrix $[\, I \mid M \,]$ is an even self-dual $[2^m, 2^{m-1}]$ code. This code is doubly-even if and only if the weight of x′ satisfies : $\omega(\mathrm{x}') \equiv 3 \mod 4$.

3 Extremal H-codes

One corollary of GLEASON's result [14] is an upper bound on the minimal distance of the even (respectively doubly-even) self-dual codes. The proof of the following theorem can be found in [17, ch. 19].

Theorem 2 *The minimum distance of a binary self-dual code with all weights divisible by* t *(and not by* $t' > t$*) is at most* d^* *where:*

- *if* $t = 2$ *then* $d^* = 2[n/8] + 2$;

- *if* $t = 4$ *then* $d^* = 4[n/24] + 4$.

A self-dual code which has minimal distance d^* *is said to be extremal. All extremal even (respectively doubly-even) codes of the same length have the same weight enumerator.*

In this section we want to prove that there are no extremal H-codes of length greater than 32. Moreover we point out that the class of H-codes is asymptotically bad. In order to do that we first determine an upper bound for the minimal distance of every H-code. This bound is given in Table 1 for $m < 14$.

Theorem 3 *Recall that* $\mathcal{A} = \mathbb{F}_2 G$, *with* $G = \mathbb{F}_{2^m}$, $m > 0$. *Let* (x) *be an* H*-code in* $\mathcal{A}$. *Then, for each* i, $i \geq 0$, *such that* $i + 2^{i-1} \leq m$, *there exists an element* y *of* (x) *whose support is a subspace of* G *of dimension* $m - i$.

Let d *be the minimal distance of* (x) *and set*

$$i^* = \max \{ i \in \mathbb{N} \mid i + 2^{i-1} \leq m \} .$$

Then $d \leq 2^{m-i^*}$.

Proof: Throughout the proof we will assume that x is an element of depth 1 of $\mathcal{A}$ and we will study the principal ideal (x). We will prove the first part of the theorem by induction on i.

The property is satisfied when $i = 0$, for any m, since each ideal of $\mathcal{A}$ contains the all-one vector $\sum_{g \in G} X^g$. If $i = 1$ the property is satisfied for any $m > 1$ since (x) contains some codewords whose support is a hyperplane of G (see Remark 1).

We suppose now that $i > 1$ and that the property is satisfied up to $i - 1$. Hence we assume that

$$(i-1) + 2^{i-2} \leq m \implies \text{there exists } \mathrm{y} \in (\mathrm{x}) \text{ such that } \mathit{supp}(\mathrm{y}) \text{ is a subspace of } G \text{ of dimension } m - (i-1) . \tag{7}$$

Now we want to prove that the property is satisfied for i.

1) We first suppose that $m = i + 2^{i-1}$. In accordance with (7) there exists an $\mathrm{y} \in (\mathrm{x})$ such that $\mathit{supp}(\mathrm{y})$ is a subspace V of G of dimension $k = m - i + 1$. Let W be the set of representatives of cosets of V. With notation of Lemma 2 we have

$$\mathrm{x} = \sum_{h \in W} \mathrm{x}(h) X^h \quad \text{where} \quad \mathrm{x}(h) \in \mathcal{P} \cap \mathbb{F}_2 V .$$

Indeed $\mathrm{y} \in (\mathrm{x})$ implies $\mathrm{yx} = 0$ which means that each $\mathrm{x}(h)$ is in $\mathcal{P}$ (from Lemma 2). Note that W contains $2^{m-k} = 2^{i-1}$ elements. Whenever $\mathrm{x}(h)$ is of depth 1, $\mathrm{x}(h)$ can be written as follows (according to (4)):

$$\mathrm{x}(h) = (X^{h'} + 1) + \mathrm{x}' \quad , \quad h' \in V \backslash \{0\} \quad , \quad \mathrm{x}' \in \mathcal{P}^2 . \tag{8}$$

Let Q be the subspace of G generated by these elements h'. Since there are 2^{i-1} elements $\mathrm{x}(h)$, the dimension of Q is less than or equal to 2^{i-1}. By hypothesis, the dimension of V

m	i^*	$d \leq$	m	i^*	$d \leq$	m	i^*	$d \leq$
2	1	2	3	1	4	4	2	4
5	2	8	6	2	16	7	3	16
8	3	32	9	3	64	10	3	128
11	3	256	12	4	256	13	4	512

Table 1: The upper bound of the minimum distance of any H-codes of length 2^m, $m \leq 13$. Notation is that of Theorem 3.

equals $2^{i-1}+1$. Hence there exists a hyperplane V' of V containing Q. Set $z = \sum_{g \in V'} X^g$ and note that $z \in \mathcal{P}^{k-1} \cap \mathbb{F}_2 V$; then we have

$$\mathrm{zx} = \sum_{h \in W} \mathrm{zx}(h) X^h \quad \text{where} \quad \mathrm{zx}(h) \in \mathcal{P}^{k-1+r} \cap \mathbb{F}_2 V \ ,$$

r being the depth of $\mathrm{x}(h)$. If $\mathrm{x}(h)$ is in $\mathcal{P}^2$ then $\mathrm{zx}(h) = 0$. Whenever $\mathrm{x}(h)$ is as in (8) we have $\mathrm{zx}' = 0$ and

$$(X^{h'}+1)z = \sum_{g \in V'} X^{g+h'} + z = 2z = 0 \ ,$$

since by construction $h' \in V'$. So $\mathrm{zx} = 0$ which is equivalent to $z \in (\mathrm{x})$. Since the support of z is the subspace V' whose dimension equals $m-i$, the property is proved for all i, when $m = i + 2^{i-1}$.

2) We suppose now that $m > i + 2^{i-1}$ and we assume that the property is satisfied up to $m-1$. In accordance with (6), there are a hyperplane H of G, an invertible element x' of $\mathbb{F}_2 H$ and $h \notin H$ such that $\mathrm{x} = X^h + \mathrm{x}'$. As $\mathrm{x}' = 1 + \mathrm{x}''$ with $\mathrm{x}'' \in \mathcal{P}$ (see (2)), $\mathrm{x} = (1 + X^h) + \mathrm{x}''$. We can choose h such that x'' has depth 1. Indeed (x) contains the half of elements of $\mathcal{P}^{m-1}$, because $\mathrm{x}\mathcal{P}^{m-1} = \{0, \sum_{g \in G} X^g\}$. So there exists a hyperplane H', different from H, which is the support of a codeword $z \notin (\mathrm{x})$. Getting h in H' (and not in H) we obtain $\mathrm{zx} = \mathrm{zx}'' \neq 0$ which yields $\mathrm{x}'' \in \mathcal{P} \backslash \mathcal{P}^2$. So we have:

$$\mathrm{x} = (X^h + 1) + \mathrm{x}'' \quad \text{where} \quad \mathrm{x}'' \in P \backslash P^2 \text{ and } \mathit{supp}(\mathrm{x}'') \subset H \ .$$

Since $\mathrm{x}'' \in \mathbb{F}_2 H$, where H is identified with the additive group of $\mathbb{F}_2^{m-1}$, there exists (by induction on m) an element y'' in (x'') whose support is a subspace of H of dimension $(m-1)-i$. Then we have $\mathrm{y}''\mathrm{x}'' = 0$ which yields that the codeword $\mathrm{y}''\mathrm{x}$ of (x) equals $\mathrm{y}''(X^h+1)$. Since $h \notin H$, the support of $\mathrm{y}''\mathrm{x}$ is a subspace of G of dimension $m-i$. That completes the proof of the first part of the theorem.

Actually we have stated an upper bound for the minimum distance d of (x). Indeed we have proved that (x) contains a codeword of weight 2^{m-i}, for any i such that $i + 2^{i-1} \leq m$. Hence $d \leq 2^{m-i^*}$, where i^* is the biggest i.
□

Corollary 2 *Set $N = 2^m$, $m > 1$. Let d_m be the minimal distance of any H-code of length N. Then the quotient d_m/N approaches zero as m approaches infinity. The class of H-codes does not meet the Gilbert-Varshamov bound : H-codes are asymptotically bad.*

Proof: In accordance with Theorem 3, $d_m \leq 2^{m-i^*}$, where i^* is the biggest i such that $i + 2^{i-1} \leq m$. Then $d_m/N \leq 2^{-i^*}$. By definition i^* approaches infinity as m approaches infinity. Then we have clearly:

$$\lim \frac{d_m}{N} = 0 \ , \quad \text{as} \ \ m \rightarrow \infty \ .$$

So we have proved that the class of H-codes does not meet the Gilbert-Varshamov bound (cf. [17, p.557]). A H-code is an $[N, N/2, d_m]$-code; then the rate k/N, where k is the dimension of the code, equals $1/2$. Let $\{ C_m \mid m > 1 \}$ be any infinite sequence of H-codes. Actually we have proved that such a sequence cannot satisfy

$$1 - K(\frac{d_m}{N}) \leq \frac{1}{2} \quad \text{where} \ \ K(s) = -s\log_2(s) - (1-s)\log_2(s-1) \ , \tag{9}$$

for all m. Indeed $K(2^{-3}) = 0.54$ and $K(2^{-4}) = 0.33$. According with Table 1, that means that (9) is not satisfied for $m \geq 12$.
□

The last corollary of Theorem 3 will give the characterization of extremal H-codes. For its proof, we must treat separately length 64.

Lemma 3 *Any H-code of length 64 contains a codeword whose support is a 3-dimensional subspace. Hence its minimal distance is less than or equal to 8.*

Proof: Let (x) be an H-code of length 64 - i.e. G is the additive group of $\mathbb{F}_{64}$. Let $z \in (x)$ such that the support of z is an hyperplane V of G ; there exists such z from Theorem 3. Since $zx = 0$, we can write x as follows (see Lemma 2):

$$x = x' + x''X^g \ , \quad g \notin V \ , \ \ x' \ \text{and} \ \ x'' \ \text{in} \ \ \mathbb{F}_2 V \cap \mathcal{P} \ .$$

According to (4) we have:

$$x' = (X^u + 1) + y' \quad \text{and} \quad x'' = (X^v + 1) + y'' \ ,$$

with $u \in V$, $v \in V$, $y' \in \mathcal{P}^2$, $y'' \in \mathcal{P}^2$. If x' (resp. x'') is in $\mathcal{P}^2$, then u (resp. v) is zero. Since x has depth 1, u and v cannot be zero together. Set

$$a' = (X^u + 1)(X^v + 1)\ x' \quad \text{and} \quad a'' = (X^u + 1)(X^v + 1)\ x'' \ .$$

If $u = 0$ or $u = v$ we take another u in V and replace it in the expression of a' and a''. So it is clear that a' and a'' are in $\mathbb{F}_2 V \cap \mathcal{P}^4$. Since V is here the additive group of $\mathbb{F}_{32}$, a' and a'' are element of the Reed-Muller code of order 1 and length 32; the support of such element (when it is a non zero element) is an affine subspace of V of dimension 4 or 5. In all cases there exists $w \neq 0$ such that the codeword

$$a = (X^u + 1)(X^v + 1)(X^w + 1)$$

has weight 8 and satisfies $aa' = 0$ and $aa'' = 0$. That yields $ax = 0$, which means $a \in (x)$, completing the proof.
□

Corollary 3 *Let* (x) *be an extremal H-code.*

(i) *If* (x) *is an even self-dual code then it is a* $[4,2,2]$ *code.*

(ii) *If* (x) *is a doubly-even self-dual code, then it is either an* $[8,4,4]$ *code, a* $[16,8,4]$ *code or a* $[32,16,8]$ *code.*

Proof: **(i)** Assume that (x) is an extremal even self-dual code. In accordance with Theorem 2, its minimal distance d equals $2^{m-2}+2$. But, from Theorem 3, we know that $d \leq 2^{m-2}$ as soon as $4 \leq m$. When $m=3$, (x) is a $[8,4,4]$ code, which is clearly doubly-even. Indeed its generator matrix is of the form

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

(see in Section 2.2). So there is only one possibility : $m=2$ and (x) is a $[4,2,2]$ code.

(ii) We now assume that (x) is an extremal doubly-even self-dual code. Then , from Theorem 2, its minimal distance d equals $4[\frac{2^{m-3}}{3}]+4$. From Theorem 3, $d \leq 2^{m-3}$ as soon as $7 \leq m$. So (x) is a code of length 2^m, with $m \leq 6$. We have seen that it is very easy to construct a doubly-even self-dual $[8,4,4]$ code. It is also easy to construct a doubly-even self-dual $[16,8,4]$ code. Indeed whenever the weight of x is 4, we are sure that (x) is doubly-even; therefore (x) has minimal distance 4. If $m=6$ then $d=12$; but, in this case, we know from Lemma 3 that d is less than or equal to 8. There is no extremal $[64,32,12]$ H-code. In the following section we will show how to construct a $[32,16,8]$ H-code.
□

4 On generators of extremal H-codes of length 32

We first recall the definition of *difference sets* in elementary abelian groups and the theorem of H.B. MANN. An overall review on difference sets can be found in [6] and [16].

Definition 2 *Let* $\{G,+\}$ *be a finite abelian group and let* D *be a subset of* G. *Then* D *is a difference set of* G *with parameter* λ *if and only if it satisfies:*

$$\texttt{card}\ (\ D \cap (h+D)\ ) = \lambda \quad \textit{for all } h \textit{ in } G\backslash\{0\}\ .$$

Theorem 4 [16] *Let* $\{G,+\}$ *be an abelian group of order* 2^k *and let* D *be a difference set of* G *with parameter* λ. *For any subgroup* V *of* G, *we denote by* $\overline{V}$ *the set* $G\backslash V$. *If* D *is equal to* $\{0\}\cup\overline{V}$, *for any* V, *we will say that* D *is a trivial difference set. If* D *is not trivial, then the parameters of* D *are:*

$$k=2t\ ,\quad \texttt{card}\ D = 2^{t-1}(2^t+\epsilon)\ \textit{ and }\ \lambda = 2^{t-1}(2^{t-1}+\epsilon)\ ,$$

where $\epsilon = \pm 1$.

From now on G is the additive group of the finite field $\mathbb{F}_{32}$.

Lemma 4 *Let* (x) *be an H-code of length* 32, *where* $x = 1 + X^h x'$, $h \notin H$, x' *invertible in* $\mathbb{F}_2 H$. *Then* (x) *is a* $[32, 16, 8]$ *code if and only if:*

(i) *The weight of* x' *is* 7 *or* 11.

(ii) *For all* $z = 1 + X^g$, $g \in H \backslash \{0\}$, *the weight of the product* zx' *is greater than* 2.

Proof: Let $[I|M]$ be the generator matrix of (x), where I is the identity matrix and $M = \{ X^u x' \mid u \in H \}$ (see Section 2). Let y in $\mathbb{F}_2 H$; then the product of the vector y by M is in fact the codeword yx'. Any codeword of (x) is of the form (y, yx') and its weight equals $\omega(y) + \omega(yx')$. By construction $yx' \neq 0$ whenever $y \neq 0$; moreover the depth of yx' is equal to the depth of y. So yx' has depth 1 for any y of weight 2.

If (x) has minimal distance 8 it is clear that (i) and (ii) are satisfied. Note that (ii) is not satisfied when $\omega(x')$ is 15.

Suppose now that (i) and (ii) are satisfied. Condition (i) implies that (x) is doubly-even. Then we must only prove that (x) does not contain a codeword of weight 4. First it is impossible to have $\omega(y) = 3$ and $\omega(yx') = 1$, since we cannot have $\omega(y) = 1$ and $\omega(yx') = 3$. Condition (ii) means that $\omega(y) = 2$ and $\omega(yx') = 2$ cannot appear, completing the proof.

□

Theorem 5 *We denote by H any hyperplane of G. Then an H-code of length* 32 *is an extremal doubly-even self-dual code (i.e. a* $[32, 16, 8]$ *code) if and only if it has a generator of the form*

$$x = 1 + X^h(1 + y) \quad , \quad h \notin H \quad , \quad y \in \mathbb{F}_2 H \ , \tag{10}$$

where the weight of $1 + y$ *is* 7 *or* 11 *and the support of* y *is a non trivial difference set of H, where H is identified with the additive group of* $\mathbb{F}_{16}$.

Proof: 1) Let (x) be an H-code such that x is of the form (10). Set $D = supp(y)$ and $z = 1 + X^g$, $g \in H \backslash \{0\}$. In accordance with Definition 2 and Theorem 4, we have

$$\mathtt{card}\ D = 6 \text{ (resp. 10)} \quad \text{and} \quad \mathtt{card}\ (D \cap (D + h)) = 2 \text{ (resp. 6)} \ .$$

Hence

$$\omega(zy) = \omega(y + X^g y) = 2\mathtt{card}\ D - 2\mathtt{card}\ (D \cap (D + h)) = 8 \ .$$

So, for all h, the weight of $z + zy$ is greater than 2. From Lemma 4, (x) is extremal.

2) Let C be an extremal H-code. According to (4), there is a generator of C of the form

$$a = (1 + X^h) + a' \ , \quad a' \in \mathcal{P}^2 \ .$$

where clearly $h \notin H$; then $a = (1 + X^h)a_1 + a_2$, where a_1 is invertible in $\mathbb{F}_2 H$ and $a_2 \in \mathcal{P}^2 \cap \mathbb{F}_2 H$. Now we obtain another generator of C:

$$x = X^h a a_1 = X^h((1 + X^h) + a_1 a_2) = 1 + X^h(1 + y) \ , \quad y \in \mathcal{P}^2 \cap \mathbb{F}_2 H \ .$$

Since C is extremal, the weight of $1+\mathrm{y}$ is 7 or 11. Now we want to prove that the support D of y is a difference set.

It is sufficient to prove that for any $\mathrm{z} = 1+X^g$, $g \in H\backslash\{0\}$, $\omega(\mathrm{zy}) = 8$. As $\mathrm{zy} \in \mathcal{P}^3 \cap \mathbb{F}_2 H$, $\omega(\mathrm{zy}) \in \{0, 8, 16\}$ (the three weights of the Reed-Muller code of order 1 and length 16). We know that $\omega(\mathrm{zy}) > 2$. Suppose that $\omega(\mathrm{zy}) = 16$. The element $\mathrm{y} \in \mathcal{P}^2$ can be identified to a boolean function f on $\mathbb{F}_{2^4}$ ($\mathrm{y} = \sum_{g\in G} f(g)X^g$) which is quadratic. Remark that zy corresponds to the function $u \rightarrow f(u) + f(u+h)$. If f is a bent function then the weight of zy is always 8 and the weight of y is 6 or 10 (see [17, p. 426-30]). If f is not a bent function, the kernel of its symplectic form is at least of dimension 2. That means that there exists $g' \in H$, $g' \neq g$, such that $\omega((1+X^{g'})\mathrm{y})$ is either 0 or 16. It is impossible to have 0; moreover

$$(1+X^g+1+X^{g'})\mathrm{y} = (X^g+X^{g'})\mathrm{y} = 2\sum_{g\in H} X^g = 0$$

proves that the value 16 is also not available. Thus we have proved that only weight 8 is possible; f is a bent function and D is a difference set.
□

Remark 3: The quadratic bent functions are completely characterized. For instance the general form of these functions is known; from [17, p. 429] we obtain the general form of the generators of the extremal $[32, 16, 8]$ H-codes. Let $\{e_1, \ldots, e_5\}$ be a basis of G and let H be the hyperplane generated by the e_i, $i \in [1, 4]$. Then a generator of an extremal H-code can be written as follows :

$$\mathrm{x} = 1 + X^{e_5}(1 + (X^{e_1}+1)(X^{e_2}+1) + (X^{e_3}+1)(X^{e_4}+1))$$

where $\omega(\mathrm{x}) = 8$. An extremal H-code with generator of weight 12 is easily obtained. For instance the following x is such generator:

$$\mathrm{x} = 1 + X^{e_5}\left(1 + (X^{e_1}+1)(X^{e_2}+1) + (X^{e_3}+1)(X^{e_4}+1) + X^{e_4}(\prod_{i=1}^{3}(X^{e_i}+1)\right).$$

5 Conclusion

In this paper we studied a special class of ideals of the algebra $\mathbb{F}_2[\{\mathbb{F}_{2^m}, +\}]$. We gave an upper bound on the minimal distance of these codes which allowed us to prove that they are asymptotically bad. We think that our bound can be improved; maybe it would be possible to find an upper bound for any ideal of the algebra which is self-dual. We conjecture that the minimum-weight codewords of the RM-code $[2^m, 2^{m-1}, 2^{(m+1)/2}]$ are important for this problem. For example, we proved in [12] that all affine invariant self-dual codes of length 128 have minimal distance 16; the value 16 is the upper bound on the minimal distance of H-codes and the minimal distance of the RM-code (for the length 128). In all cases there is a codeword of weight 16 whose support is a vector-space of dimension 4.

The complete classification of all binary, self-dual, doubly-even $[32, 16]$ codes is due to CONWAY and PLESS [13]; they proved that there are five non-equivalent extremal codes (another proof was given later in [15]). Two of the five codes are well-known extended cyclic

codes; that is the Reed-Muller code and the quadratic residue code. There are three other extremal codes of length 32, namely the *code F*, the *code G* and the *code U*. We proved that the generator of any extremal $[32, 16]$ H-code is completely defined by a quadratic bent function on $\mathbb{F}_{16}$. These functions correspond to the codewords of weight 6 or 10 of the RM-code $R(2, 4)$. All codewords of weight 6 (resp. 10) are equivalent. So there are at most two non-equivalent H-codes; from recent explorations it seems that all H-codes are equivalent to the code G. A full explanation of these facts in a more general context will be presented in a forthcoming paper.

The complete coset weight distributions of the five extremal $[32, 16]$ codes are given in [7] (see also [3] for the quadratic residue code).

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